Integration – using the ‘area so far’ function

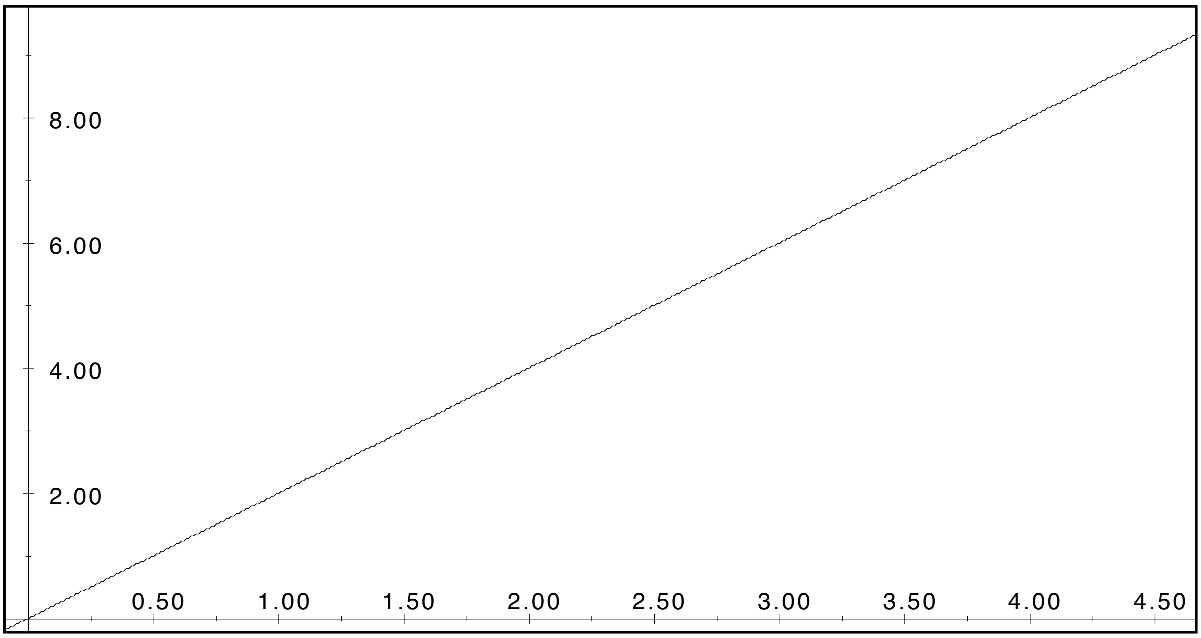
Level
Upper secondary

Mathematical Ideas
Area under a curve, integration, collecting and modelling data

Description and Rationale
The area under a curve can be a very useful mathematical tool. When students begin studying integral calculus methods such as the trapezoidal, the Monte Carlo and upper and lower rectangle methods are used to determine the area under a curve. These are estimates that can be very accurate under the right conditions (large sample, small increments etc..) but they are still estimates and thus do not provide exact answers. Further they are often cumbersome and time consuming in their calculation and do not provide general relationships. This activity attempts to explore the possibility of finding a mathematical relationship between a function and the area under the curve defined by the function.

One way of investigating the relationship between two quantities is to collect some information, make a conjecture and then check its validity. To do this it is best to start with a very simple example. A linear function is such an example.

Consider the function $y = 2x$, the graph of which appears below.



Students would be asked to investigate the area under this curve upward from $x = 0$.

Students determine the area under the curve between x = 0 and a range of x–values greater than zero, this information is then recorded and modelled.

Example

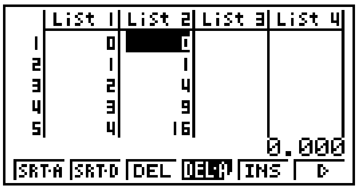
Between $x = 0$ and $x = 2$ $\qquad A = \frac{1}{2} \cdot 2 \cdot 4 = 4$ square units.

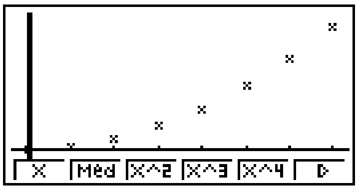
Between $x = 0$ and $x = 4$ $\qquad A = \frac{1}{2} \times 4 \times 8 = 16$ square units.

A sample of the possible data that may be generated is recorded in the table below.

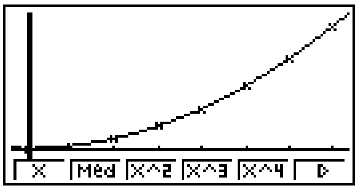
X	0	1	2	3	4	5	6	7
AREA	0	1	4	9	16	25	36	49

This data can then be placed into lists and a scatter plot produced.





Students would then be expected to model this data in an appropriate way. Students may observe the data collected and see the model would be "area so far" = x^2. Others may note the shape and predict a quadratic model. A third method involves using the built in quadratic regression modelling tool.

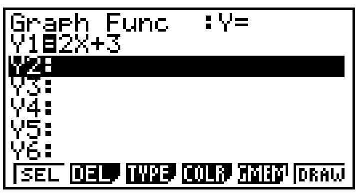


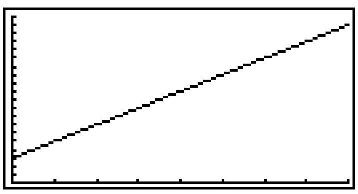
Based on the information generated above students would be expected to produce a conjecture along the lines of:

"the area under the curve from x = 0" function for the function $y = 2x$ is $A = x^2$.

The area under a curve can easily be determined using the calculator in either the GRAPH mode (using the commands in G-solv) or the RUN mode (using commands in OPTN and VARS).

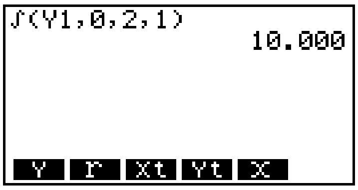
Examples of each, for the function $y = 2x + 3$, are shown below.



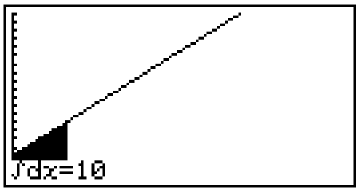


The examples below find the area between $x = 0$ and $x = 2$ for $y = 2x + 3$

In the RUN mode the integral sign can be found under OPTN and CALC (F4). The function Y can be found using VARS and GRPH (F4). The lower and upper terminals are defined along with a number which relates to the tolerance method employed by the calculator, 1 is the most accurate.



In the GRAPH mode the integral sign is found in the G-Solv menu. Sideways arrows and EXE are used to define the lower and upper limits.

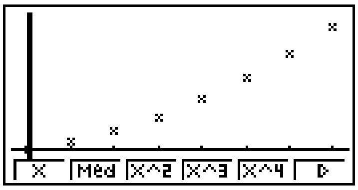


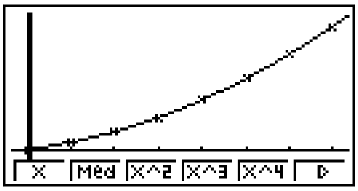
Students may then be asked to use one of the methods above to collect data and model the rule for “the area under the curve so far up from $x = 0$” for the function $y = 2x+3$.

A possible data set that may be generated by the students is shown below.

X	0	1	2	3	4	5	6	7
AREA	0	4	10	18	28	40	54	70

The resulting scatter plot and model could be determined using one of the methods described above. The model is $y = x^2 + 3x$.





So the “area under the curve so far from x=0” for the function $y = 2x+3$ is $A = x^2+3x$.

Students would then be asked to make a general conjecture about the relationship between these two functions. It is hoped that they would begin to see the idea of anti-differentiation of polynomials as they would previously have studied the rules for differentiation.
To verify the conjecture members of the class could each be given a function to investigate, with the results summarised in a table such as shown below. This may generate much class discussion and collaboration between students as they develop and discover the relationship between the functions. The examples chosen, at this stage, must have areas that are restricted to being above the x- axis.

Group results

Original function	“Area so far” function
$2x + 1$	$x^2 + x$
$6x + 4$	$3x^2 + 4x$
$-3x + 30$	$-1.5x^2 + 30x$

Beyond Linear functions

As we know mathematical models are not limited to just linear functions.
The motion of a body thrown upwards under gravity produces a parabolic position-time graph. The mathematical representation of this is:

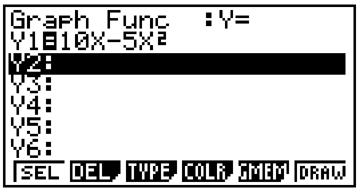
$$s = ut - \frac{1}{2}gt^2$$

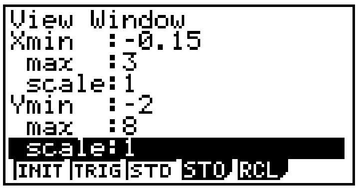
where s is the position (m), u is the initial velocity (m/s), t is the time (s) and g is the acceleration due to gravity (m/s^2). We need to check whether our conjecture from our work with linear functions still holds with a more complex function.

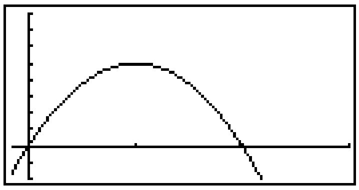
Consider a body thrown upwards at 10m/s (u), assuming that gravity (g) is $10m/s^2$.

The position is given by $s = 10t - 5t^2$

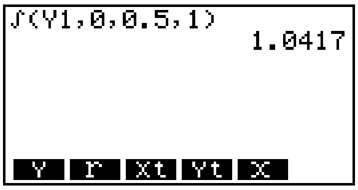
Students could be asked to produce a plot, collect data relating to the "area so far" for a range of x-values and then complete an analysis as before. Students should be encouraged to take into account the conclusions drawn from the linear case.
The view window is chosen to allow x values for the integral to easily be selected. The window uses the fact that the screen is 126 pixels wide.

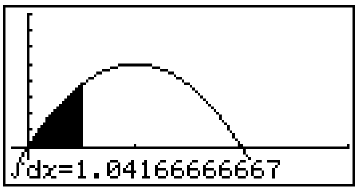






The examples below involve choosing a lower bound of 0 and an upper bound of 0.5.

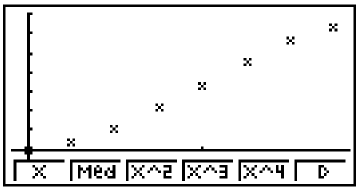


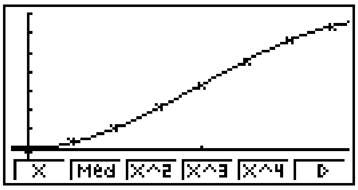


A set of data, such as that shown below, should be generated.

X	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75
AREA	0	0.287	1.042	2.109	3.333	4.557	5.625	6.380

The scatter plot suggests a cubic function, using the ideas explored previously a model must be fitted.





Students would be expected to explain clearly how the information gained from the linear cases relates to that gained from the more complex function. A good student should be able to clearly articulate the development of their understanding of the "area so far from x = 0" function.

Students understanding could be quickly checked by a series of examples set out in a table such as that below.

Use what you have learned to write down the "area so far from x = 0" functions for each of the following:

Original function	"Area so far from x = 0" function
$y = 3x - 2$	
$y = -3x^2 + 2x$	
$y = 4x - x^2$	
$y = 2x^2 - 3x + 4$	
$y = x^2 + 2$	

Limitations

You have found a rule for the "area so far" for most polynomials. This process is known as integration and the "area under the curve" function is known as the integral. You may also have noticed that this is the same as deriving backwards. For this reason this process is often called anti-differentiation and the resulting function called the anti-derivative.

This rule has proved fairly effective for a range of functions but we need to be aware of its limitations.

Students may have noticed that we have only investigated the "area so far' functions that start at $x = 0$. What alterations to this method do we need to make to find the area under a curve between two non-zero values, eg the area under the curve $s = 10t - 5t^2$ between $t = 2$ and $t = 4$?

Students could be asked to investigate this problem and present their findings to the rest of the class. They would be expected to write down what they have found and provide evidence that their solution is correct.

A further extension would be to investigate situations where the graph crosses the axis within the domain explored. The ideas of positive and negative areas and the separation of a function into regions for determining the area would then be explored.