Kepler's Law

Level
Upper secondary

Mathematical Ideas
Modelling data, power variation, straightening data with logarithms, residual plots

Description and Rationale
Many traditional mathematics problems have been given a new life with the advent of the graphics calculator. The time spent in completing repetitive calculating tasks can be greatly reduced with the use of a graphics calculator, allowing for a greater focus on the important mathematics involved in the task. This activity allows students to analyse historical data to determine a famous mathematical relationship. The activity may be used at a variety of levels and allows for a variety of approaches. A data set relating periods and orbital radii for a number of planets in our solar system is provided, along with the general accepted form of the model. The model is determined and compared to its accepted form, along with a view of a residual plot to confirm the acceptability of the model.

Johann Kepler is a very famous man in the history of astronomy and mathematics. He used data from observations of planetary orbits to show that these motions were not random, that they in fact obeyed certain mathematical laws, and that these laws could be written in algebraic form. The data below uses the earth as the base unit, so times and distances are given as multiples of 1 earth year, and the average radius of our Earth's orbit around the sun.

Some of his observational data are given below:

Planet	**Orbital period T**	**Orbital radius R**
Mercury	0.241	0.387
Venus	0.615	0.723
Earth	1.000	1.000
Mars	1.881	1.542
Jupiter	11.862	5.202
Saturn	29.457	9.539

Kepler showed that a power model could be used to describe this data.

i.e. $T = aR^b$ where a and b are constants.

The task is to find appropriate values for a and b, and to justify the procedure you use.
A follow up task involves reorganising the solution into the traditional expression of the law.

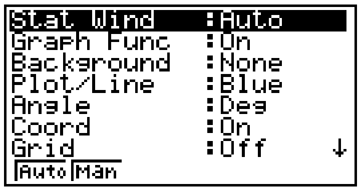
This law is usually stated in the form $T^m = k R^n$, where m and n are whole numbers. Determine the simplest values of m and n and complete Kepler's law.

"…The __________________ of a planet's orbital period (T in years) is

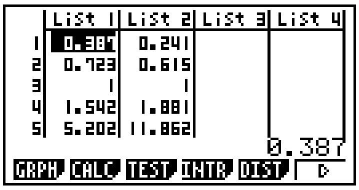
directly proportional to the __________________ of the average distance

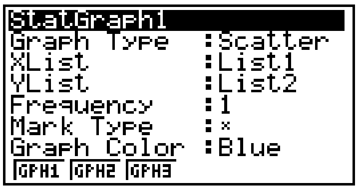
(R in metres) from the sun…"

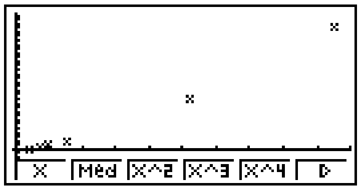
For an easy scatter plot the AUTO window needs to be defined. This is done in the SET UP mode (SHIFT then MENU).



The data needs to be entered in the STAT mode and plotted.



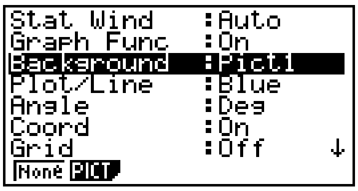




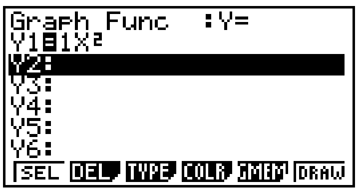
The formulation of an appropriate model may be completed in a number of ways.

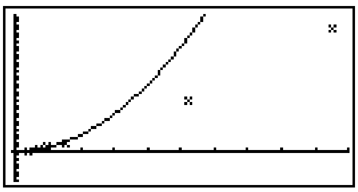
Students may be asked to develop the model by appropriately selecting and adjusting values for the two constants *a* and *b* until a satisfactory model is produced.

The first step is to place the scatter plot in the background of the graphing screen. A picture is taken with the scatter plot on the screen via the OPTN then PIC menus. This is placed in the background by going into the SET UP (SHIFT then MENU) and making the saved picture the background



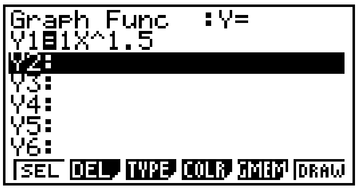
A starting model needs to be created as shown. The curved nature of the scatter plot suggesting a starting model as shown.

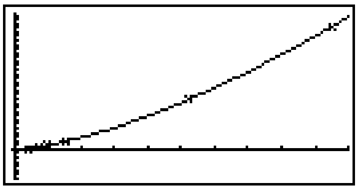




The scatter plot produced suggests that the power of 2 is too large, a value less than 2 (1.5) needs to be tested.

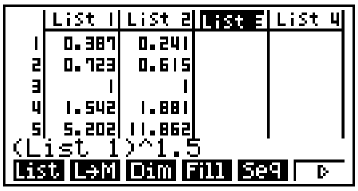
The new model and resulting scatter plot are shown below.



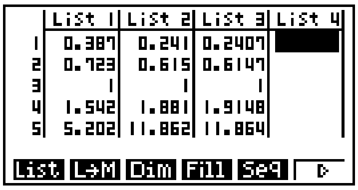


This would appear to be suitable model of the data provided. The value of *a* can be analysed by returning to the lists and completing the operation of the rule on the values in List 1 and then comparing List 2 (actual data) and List 3 (predicted values).

To operate on a list of numbers the list name is highlighted. The available list commands are found via the OPTN button then LIST menu.

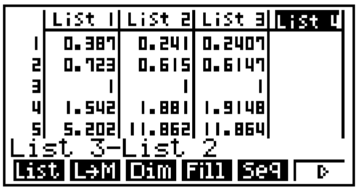


The resulting table of actual and predicted data can then be compared.

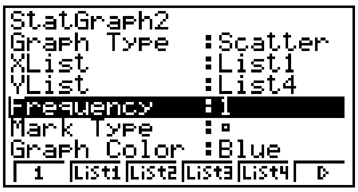


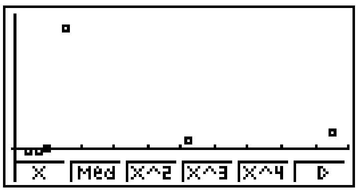
A method of determining the quality of the model is to do a Residual plot. This involves looking for a pattern with the differences between the actual data and the results predicted by the model. A plot where the residuals appear to be randomly distributed supports the proposed model.

The residuals may be calculated by defining the operation as shown.



The set up of the scatter plot and the resulting plot are shown below.





The variations from the proposed model are small in size although not completely randomly distributed.

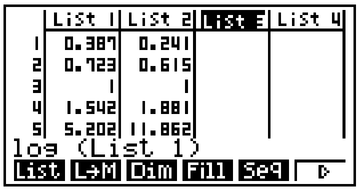
A second approach to this task would involve attempting to straighten the data using logarithms. Natural logarithms or logarithms to the base 10 may be used.

If the proposed model is $T = a R^b$ then :

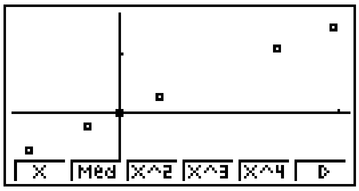
$$\log T = \log (a R^b)$$
$$\log T = \log a + \log R^b$$
$$\log T = \log a + b \log R$$
$$\log T = b \log R + \log a$$

This is in the general form of a straight line (y = mx + c) and so if the model is appropriate the data above may be modelled by a straight line if plotted as log T vs log R. The gradient of this plot gives the value for b, while the y-intercept provides the value of log a.

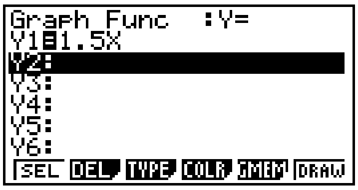
The commands to find the logarithms of List 1 are shown. The commands are accessed via the OPTN button and LIST menu.

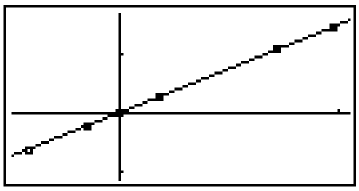


The scatter plot of log T vs log R is shown.



A model may be fitted as before or by using the regression potential of the calculator.





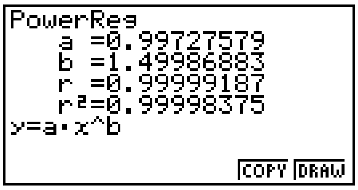
This model must now be reconstructed into the required form.

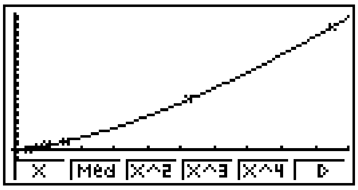
$$\log T = 1.5 \log R + 0$$
$$\log a = 0, \quad a = 1, \ n = 1.5$$
$$T = 1 . R^{1.5}$$

The third method uses the regression model potential of the calculator, while this is the least mathematically rich option for the student there are times when this method would be used. The added bonus is that the residual analysis can easily be completed.

In the STAT mode the residuals can be stored in a list.
In the SET UP mode (SHIFT then MENU) the residuals have been placed into list 3.

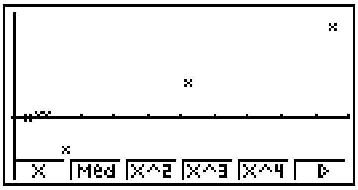
The scatter plot can be produced, the power model (Pwr) generated and the model placed over the data.





The calculator has placed the residuals placed into List 3.

A scatter plot of the residuals (List 3) vs the orbital radii (List 1) can easily be produced for investigation.



A model of the form $T = R^{1.5}$ has been generated by each of the methods employed. The second part of the task involved determining the general form of Kepler's law.

If the power is 1.5 then if both sides are squared the model becomes $T^2 = R^3$. The general statement of Kepler's law becomes:

> "…The **square** of a planet's orbital period (T in years) is directly proportional to the **cube** of the average distance (R in metres) from the sun…"

This is in fact a result that can be determined from Newton's Law of Gravitation. Newton's Law states that the force between two objects is directly proportional to the product of the two masses (m_1 and m_2) and inversely proportional to the square of their separation (R).

$F = \frac{Gm_1m_2}{R^2} = ma$ where G is the Universal Gravitation constant, 6.67×10^{-11} Nm^2kg^{-2}

For orbiting bodies this acceleration a is $\frac{v^2}{R}$, $v = \frac{2\pi R}{T}$ therefore $a = \frac{4\pi^2 R}{T^2}$

Subbing this into the above equation results in $\frac{R^3}{T^2} = \frac{GM}{4\pi^2}$, where M is the mass of the central body (the sun).

The right hand side of the above expression is the constant of proportionality, dependent only on the mass of the central body of the system.